

First-Order System \mathcal{LL}^* (FOSLL *) for Maxwell's Equations

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Web Page

<http://amath.colorado.edu/faculty/tmanteuf/papers.html>

Motivation

Numerical approximation of a physical phenomenon requires a series of distinct but interrelated steps:

Steps	Issues
Physical Model	Reasonableness
Mathematical Model	Well-posedness
Discrete Model	Accuracy
Numerical Solution	Computational Cost
Software	Machine Architecture

- Decisions at one level affect levels below
- For best results, all levels must be considered simultaneously

FOSLS Approach: Recast mathematical model to yield

- Systematic Approach
- Optimal Approximation
- Efficient Numerical Solution

Outline

FOSLS = first-order system least squares

FOSLL = first-order system least LL^* norm*

- I. Motivation
- II. Problem Description
- III. FOSLS Formulation
 - a. Smooth Coefficients and Regular Domains
 - b. Numerical Results
 - c. Discontinuous Coefficients and Irregular Domains
 - d. Numerical Results
- IV. FOSLL* Formulation
 - a. Discontinuous Coefficients
 - b. Irregular Domains
 - c. Numerical Results
- V. Summary

FOSLS Process

- Establish equivalent first-order system by introducing new dependent variables
- Add constraint equations to yield possibly overdetermined but consistent system
- Create a least-squares functional by taking either an L^2 (or H^{-1}) norm (weighted) of each equation
- Prove ellipticity (coercivity and continuity) of functional in a meaningful norm
 - L^2 Functional $\rightarrow H^1$ equivalence
 - H^{-1} Functional $\rightarrow L^2$ equivalence
- Goals:
 - Good approximation properties
 - Fast numerical solution
 - Independence of problem parameters

Advantages of FOSLS

- Minimization Principle

- Ellipticity in Meaningful Norm
- Discretization by Subspace Restriction
- Convergence through Approximation (Cea's Lemma)

- Well Posed Functional \Rightarrow well posed discrete problem

- Simple finite elements
- No LBB condition
- No staggered grids
- Simple boundary conditions
- SPD linear system
- Condition $O(h^{-2})$

- Minimum Functional Value = 0

- Functional provides a computable error measure
- Adaptive Refinement straightforward
- Nonconforming Finite Elements straightforward

Advantages of FOSLS

Advantages of H^1 Ellipticity

- H^1 error bounds for all variables
- Optimal multigrid convergence

Advantages of L^2 Ellipticity

- L^2 error bounds for all variables
- Reduced regularity requirements
- Optimal convergence standard iterative methods

Disadvantages of FOSLS

- More dependent variables
 - Maybe they are the real objective
 - Possibly fewer unknowns
- H^1 ellipticity requires increased regularity
 - Multigrid convergence proof: $H^1(\Omega)$ sufficient
 - $(H(\text{div}; \Omega) \cap H(\text{curl}; \Omega))$ sufficient also
 - H^{-1} Functional available
- Conservation not automatic
 - Often possible
 - Not always necessary
- Correct formulation often elusive

Maxwell's Equations: Eddy Current Approximation

H Magnetic Field

E Electrical Field

B Magnetic Flux Density

J Current Density

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad \text{Faraday's Law}$$

$$\nabla \times \mathbf{H} - \mathbf{J} = 0 \quad \text{Ampere's Theorem}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{Initial Condition}$$

$$\nabla \cdot \mathbf{J} = 0$$

Constitutive Relations

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{Ohm's Law}$$

Maxwell's Equations: Eddy Current Approximation

Reduced Form

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} - \sigma \mathbf{E} = 0$$

$$\nabla \cdot \mu \mathbf{H} = 0$$

$$\nabla \cdot \sigma \mathbf{E} = 0$$

Boundary Conditions

$\mathbf{n} \times \mathbf{E}$, $\mathbf{n} \cdot \mathbf{H}$ prescribed on Γ_N consistently

$\mathbf{n} \times \mathbf{H}$, $\mathbf{n} \cdot \mathbf{E}$ prescribed on Γ_D consistently

Maxwell's Equations in 2-D

Assumptions

$$\mathbf{H} = \begin{pmatrix} 0 \\ 0 \\ \eta_3(x, y) \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} e_1(x, y) \\ e_2(x, y) \\ 0 \end{pmatrix}$$

Equations ($\nabla^\perp = \text{rot } \mathbf{E}$)

$$\mu \frac{\partial \eta_3}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\nabla^\perp \eta_3 - \sigma \mathbf{E} = 0$$

$$\nabla \cdot \sigma \mathbf{E} = 0$$

Boundary Conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{E} \text{ prescribed on } \Gamma_N &\text{ implies } \mathbf{n} \cdot \nabla \eta_3 \\ \eta_3 \text{ prescribed on } \Gamma_D &\text{ implies } \mathbf{n} \cdot \mathbf{E} \end{aligned} \tag{1}$$

FOSLS Formulation

Backward Euler in Time

$$\frac{\mu}{\delta t} \eta_3 + \nabla \times \mathbf{E} = \frac{\mu}{\delta t} \eta_{old}$$

$$\nabla^\perp \eta_3 - \sigma \mathbf{E} = 0$$

$$\nabla \cdot \sigma \mathbf{E} = 0$$

FOSLS Functional

$$G(\eta, \mathbf{E}; \eta_{old}) := \left\| \frac{\mu}{\delta t} \eta + \nabla \times \mathbf{E} - \frac{\mu}{\delta t} \eta_{old} \right\|^2 + \|\nabla^\perp \eta - \sigma \mathbf{E}\|^2 + \|\nabla \cdot \sigma \mathbf{E}\|^2$$

Ellipticity Define $H_{DC_\sigma} := H(div, \sigma) \cap H(curl)$ with norm

$$\|\mathbf{E}\|_{DC_\sigma}^2 := \|\mathbf{E}\|^2 + \|\nabla \cdot \sigma \mathbf{E}\|^2 + \|\nabla \times \mathbf{E}\|^2$$

Coercive and Continuous

$$c_0 (\|\eta\|_1^2 + \|\mathbf{E}\|_{DC_\sigma}^2) \leq G(\eta, \mathbf{E}; 0) \leq c_1 (\|\eta\|_1^2 + \|\mathbf{E}\|_{DC_\sigma}^2)$$

Discrete Equations

Finite Element Spaces V^h and W^h .

Find $\eta^h \in V^h$ and $\mathbf{E}^h \in W^h$:

$$(\eta^h, \mathbf{E}^h) = \arg \min_{(V^h, W^h)} G(\eta, \mathbf{E}; \eta_{old})$$

Weak form

Find $\eta^h \in V^h$ and $\mathbf{E}^h \in W^h$:

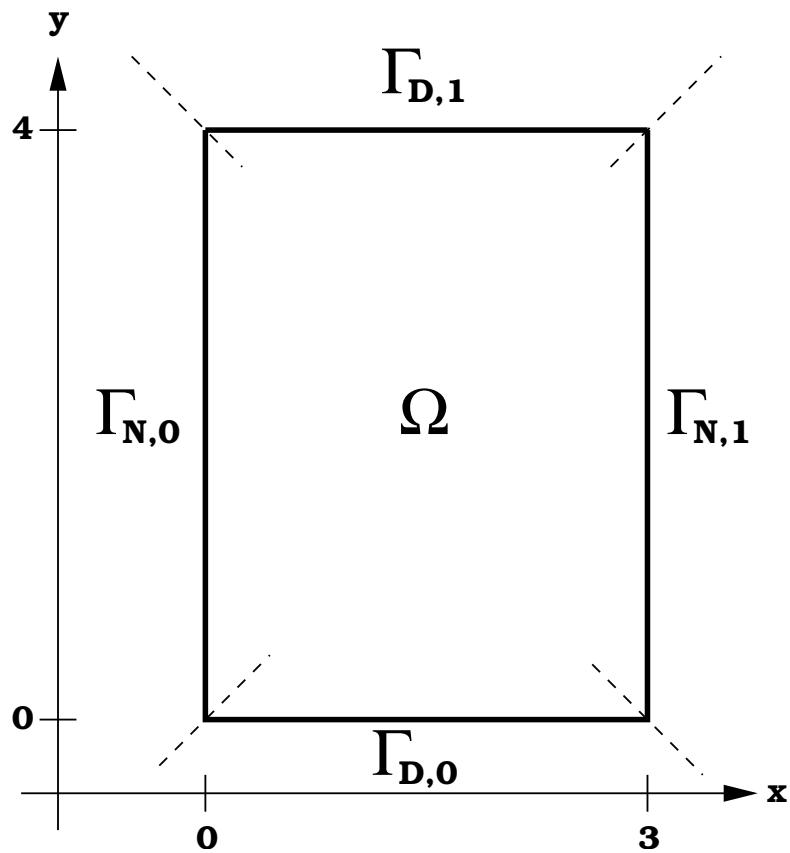
$$\left\langle \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times \\ \nabla^\perp & -\sigma \\ 0 & \nabla \cdot \sigma \end{bmatrix} \begin{pmatrix} \eta^h \\ \mathbf{E}^h \end{pmatrix}, \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times \\ \nabla^\perp & -\sigma \\ 0 & \nabla \cdot \sigma \end{bmatrix} \begin{pmatrix} g^h \\ \mathbf{F}^h \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \frac{\mu}{\delta t} \eta_{old} \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times \\ \nabla^\perp & -\sigma \\ 0 & \nabla \cdot \sigma \end{bmatrix} \begin{pmatrix} g^h \\ \mathbf{F}^h \end{pmatrix} \right\rangle$$

for every $g^h \in V^h$ and $\mathbf{F}^h \in W^h$

Discrete Error Bound:

$$\|\eta - \eta^h\|_1 + \|\mathbf{E} - \mathbf{E}^h\|_{DC_\sigma} \leq Ch^k (\|\eta\|_{1+k} + \|\mathbf{E}\|_{1+k})$$

Problem I



$$\begin{aligned} \mathbf{n} \cdot \mathbf{E} &= f'(x), \quad \eta = f(x) && \text{on } \Gamma_{D,1} \\ \mathbf{n} \cdot \mathbf{E} &= 0, \quad \eta = 0 && \text{on } \Gamma_{D,0} \\ \mathbf{n} \times \mathbf{E} &= 0 && \text{on } \Gamma_{N,i} \end{aligned}$$

with

$$f(x) = 1 - \cos(2\pi x/3)$$

- Steady State ($\eta_{old} = 0$)
- Smooth Coefficients ($\sigma = 1$)
- Regular Boundary Conditions
- $H_{DC_\sigma} = H(div, \sigma) \cap H(curl) = (H_1)^2$

Numerical Results for Problem I

- Steady State
- Smooth Coefficients ($\sigma = 1$)
- Regular Boundary Conditions
- $H_{DC_\sigma} = H(\text{div}, \sigma) \cap H(\text{curl}) = (H_1)^2$

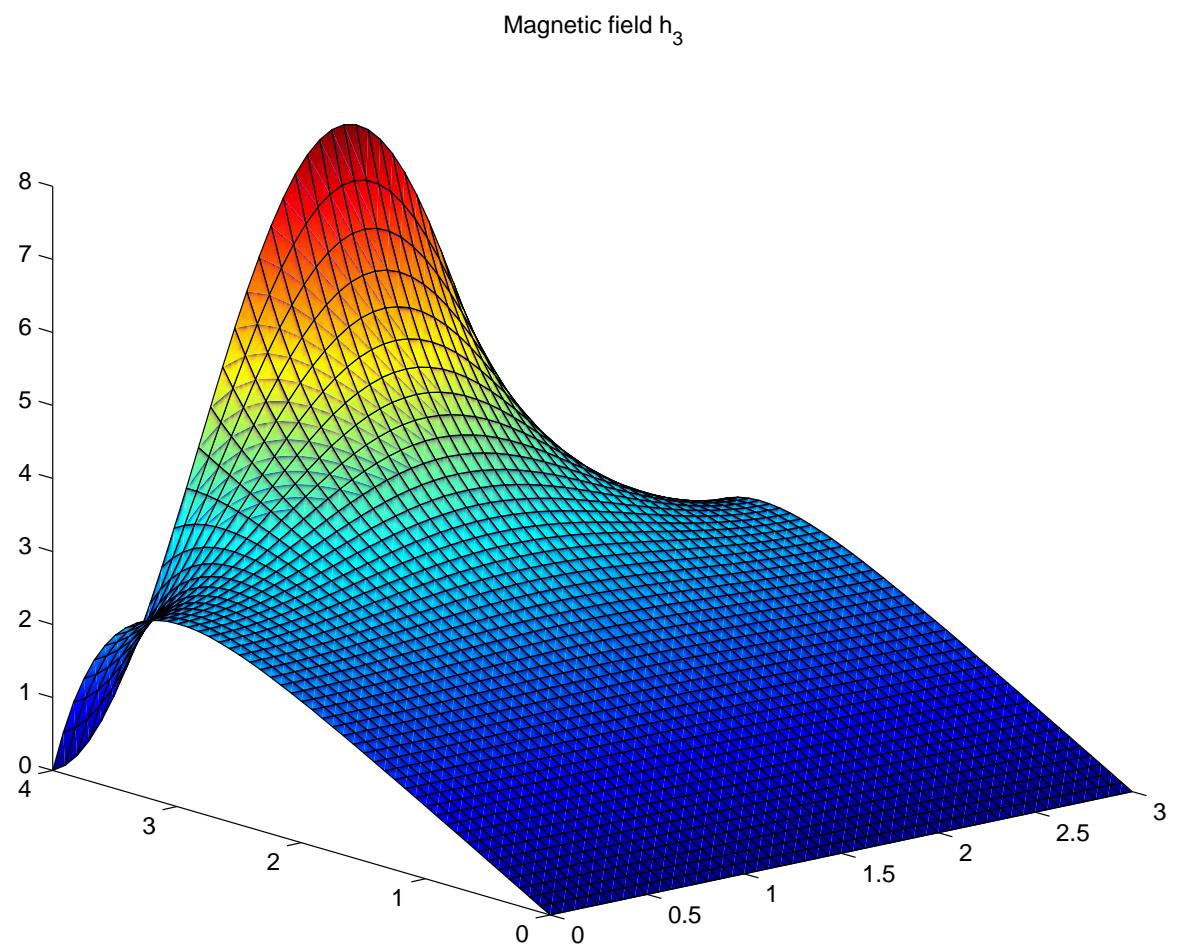
Convergence results :

L/h	MG(V)	MG(W)	$G(\eta^h, \mathbf{E}^h)$	Reduction
12	0.48	0.084	21.038	$2^{-2} = 0.25$
24	0.42	0.033	5.321	0.2538
48	0.46	0.019	1.334	0.2507
96	0.47	0.012	0.334	0.2503
192	0.46	0.009	0.084	0.2499

- Rectangular Mesh
- Bilinear Finite-elements
- Multigrid V(1,1) and W(1,1)-Cycle

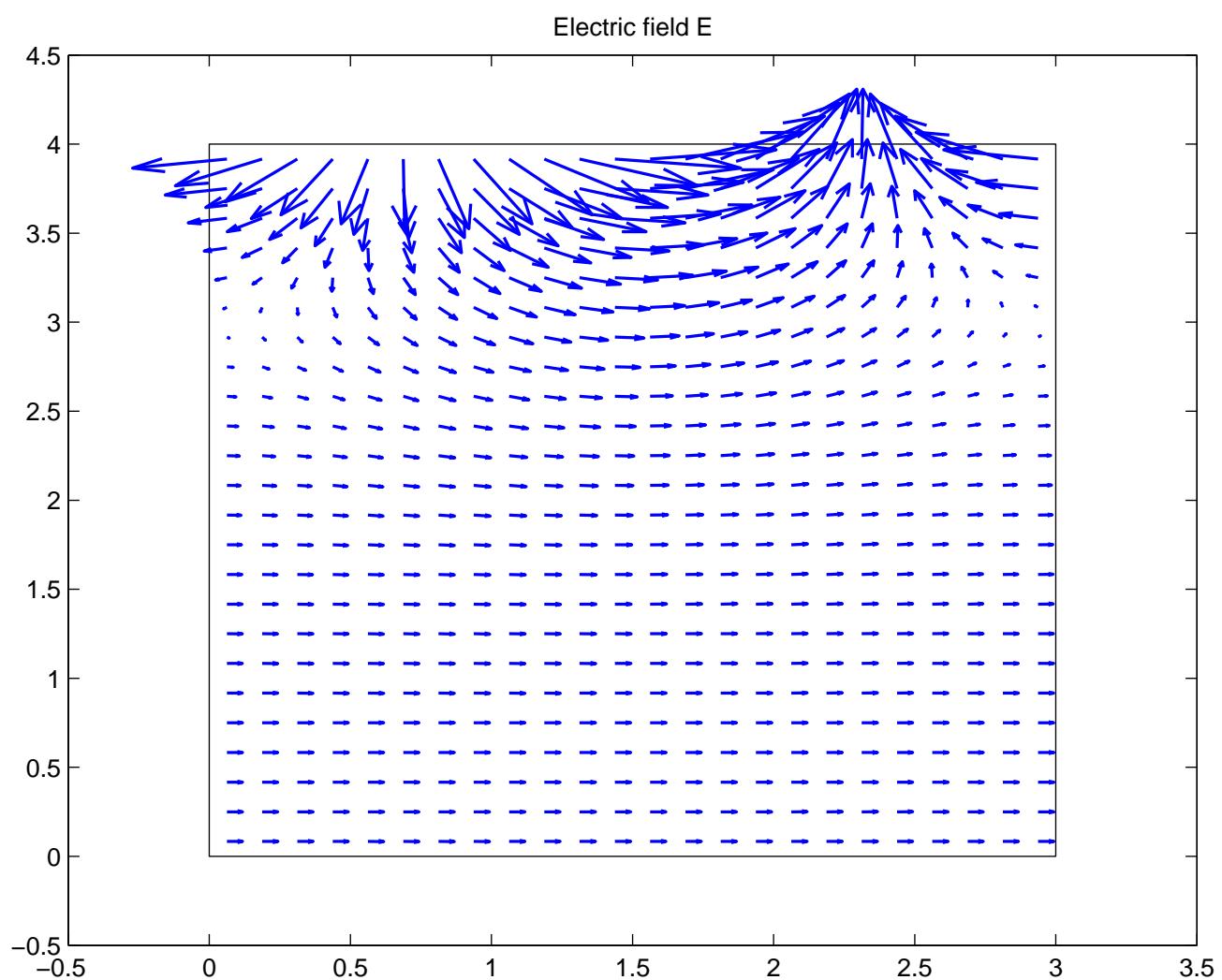
Numerical Results

Magnetic field η^h :

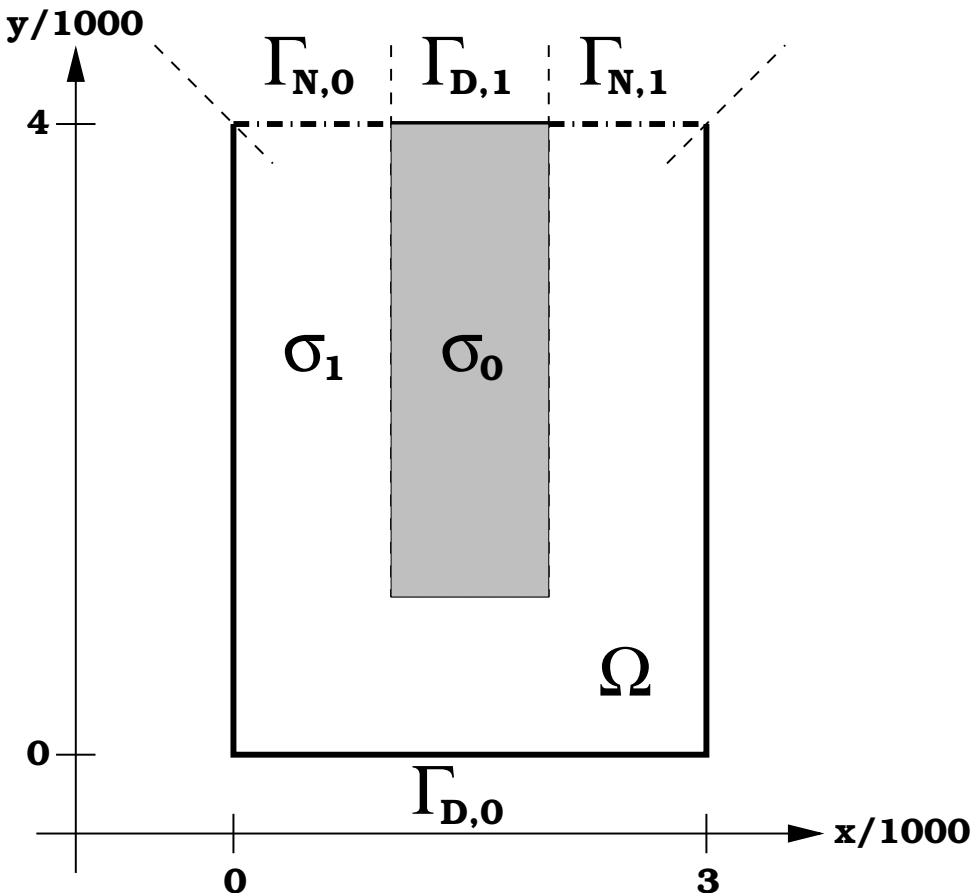


Numerical Results

Electric field \mathbf{E}^h :



Problem II



$$\begin{aligned} \mathbf{n} \cdot \mathbf{E} = 0, \quad \eta = 1 && \text{on } \Gamma_{D,1} \\ \mathbf{n} \cdot \mathbf{E} = 0, \quad \eta = 0 && \text{on } \Gamma_{D,0} \\ \mathbf{n} \times \mathbf{E} = 0 && \text{on } \Gamma_{N,i} \end{aligned}$$

- Steady State
- Discontinuous Coefficients ($\sigma_0 = 1, \quad \sigma_1 = 6.33 \cdot 10^7$)
- Irregular Boundary Conditions
- $H_{DC_\sigma} = H(\operatorname{div}, \sigma) \cap H(\operatorname{curl}) \neq (H_1)^2$

Numerical Results for Problem II

- Steady State
- Smooth Coefficients $(\sigma_0 = \sigma_1 = 1.0)$
- Irregular Boundary Conditions
- $H_{DC_\sigma} = H(\operatorname{div}, \sigma) \cap H(\operatorname{curl}) \neq (H_1)^2$

Convergence results :

L/h	MG(V)	MG(W)	$G(\eta^h, \mathbf{E}^h)$	Reduction
12	0.23	0.073	32.352	
24	0.25	0.036	9.325	0.2882
48	0.24	0.019	3.210	0.3442
96	0.25	0.013	1.654	0.5153
192	0.32	0.009	1.261	0.7624

- Rectangular Mesh
- Bilinear Finite-elements
- Multigrid V(1,1) and W(1,1)-Cycle

Alternatives

- Local Refinement Does Not Work!
- Augmented Finite Element Spaces
- Weighted L^2 -Norm Functional Weight function $w = r^\alpha$

$$G_w(\eta, \mathbf{E}; \eta_{old}) := \left\| \frac{\mu}{\delta t} \eta + \nabla \times \mathbf{E} - \frac{\mu}{\delta t} \eta_{old} \right\|_w^2 + \|\nabla^\perp \eta - \sigma \mathbf{E}\|_w^2 + \|\nabla \cdot \sigma \mathbf{E}\|_w^2$$

- H^{-1} -Norm Functional

$$G_{-1}(\eta, \mathbf{E}; \eta_{old}) := \left\| \frac{\mu}{\delta t} \eta + \nabla \times \mathbf{E} - \frac{\mu}{\delta t} \eta_{old} \right\|_{-1}^2 + \|\nabla^\perp \eta - \sigma \mathbf{E}\|_{-1}^2 + \|\nabla \cdot \sigma \mathbf{E}\|_{-1}^2$$

- FOSLL* Formulation

FOSLL* Formulation

FOSLS Functional $\mathcal{L}\mathcal{U} = \mathcal{F}$

Find \mathcal{U} that minimizes

$$G(\mathcal{U}) = \langle \mathcal{L}\mathcal{U} - \mathcal{F}, \mathcal{L}\mathcal{U} - \mathcal{F} \rangle$$

FOSLS Weak Form Primal Problem

Find \mathcal{U} such that

$$\langle \mathcal{L}U, \mathcal{L}V \rangle = \langle \mathcal{F}, \mathcal{L}V \rangle \quad \forall V$$

FOSLL* Formulation $\mathcal{L}\mathcal{L}^*\mathcal{W} = \mathcal{F}$ or $\mathcal{L}^*\mathcal{W} = \mathcal{U}$

Find \mathcal{W} that minimizes

$$G_*(\mathcal{W}) := \|\mathcal{L}^*\mathcal{W} - \mathcal{U}\|^2$$

Find \mathcal{W} such that

$$\langle \mathcal{L}^* \mathcal{W}, \mathcal{L}^* \mathcal{V} \rangle = \langle \mathcal{U}, \mathcal{L}^* \mathcal{V} \rangle = \langle \mathcal{F}, \mathcal{V} \rangle$$

Existence for Dual Problem

How do we know there is a \mathcal{W} such that $\mathcal{L}^*\mathcal{W} = \mathcal{U}$?

Define:

$\mathcal{L} : \mathcal{D} \rightarrow V$ with \mathcal{D} dense in Hilbert Space V

and

$\mathcal{L}^* : \mathcal{D}^* \rightarrow V$ with \mathcal{D}^* dense in Hilbert Space V .

Lemma 1:

Assume that \mathcal{D} and \mathcal{D}^* are compactly embedded in V and \mathcal{L} and \mathcal{L}^* are continuous on \mathcal{D} and \mathcal{D}^* respectively. Then \mathcal{L} is 1-to-1 and onto and coercive $\Leftrightarrow \mathcal{L}^*$ is 1-to-1 and onto and coercive

Lemma 2:

Under the same hypotheses, if \mathcal{L} is coercive on \mathcal{D} , and \mathcal{L}^* is 1-to-1, then \mathcal{L}^* is also coercive and onto. This implies $\mathcal{L}^*\mathcal{W} = \mathcal{U}$ has a unique solution in \mathcal{D}^* .

FOSLL* Formulation for Maxwell

System Form

$$\begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times \\ \nabla^\perp & -\sigma \\ 0 & \nabla \cdot \sigma \end{bmatrix} \begin{pmatrix} \eta \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\delta t} \eta_{old} \\ \mathbf{0} \\ 0 \end{pmatrix}$$

Slack Variable $q = 0$

$$\begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\nabla \\ 0 & \nabla \cdot \sigma & 0 \end{bmatrix} \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\delta t} \eta_{old} \\ \mathbf{0} \\ 0 \end{pmatrix}$$

Boundary Conditions

$$\boldsymbol{\tau} \cdot \mathbf{E}, \quad q \quad \text{prescribed on } \Gamma_N$$

$$\mathbf{n} \cdot \mathbf{E}, \quad \eta \quad \text{prescribed on } \Gamma_D$$

FOSLL* Formulation for Maxwell

Notation $\mathcal{LU} = \mathcal{F}$

$$\mathcal{LU} = \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\nabla \\ 0 & \nabla \cdot \sigma & 0 \end{bmatrix} \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix}$$

Homogeneous Boundary Conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{E} &= 0, & q &= 0 && \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{E} &= 0, & \eta &= 0 && \text{on } \Gamma_D \end{aligned}$$

Adjoint Operator

$$\mathcal{L}^* \mathcal{W} = \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} \eta_* \\ \mathbf{E}_* \\ q_* \end{pmatrix}$$

Adjoint Boundary Conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{E}_* &= 0, & q_* &= 0 && \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{E}_* &= 0, & \eta_* &= 0 && \text{on } \Gamma_D \end{aligned}$$

Primal Problem

$$\mathcal{LU} = \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\nabla \\ 0 & \nabla \cdot \sigma & 0 \end{bmatrix} \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{0} \\ 0 \end{pmatrix} = \mathcal{F}$$

Primal Boundary Conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{E} &= 0, & q &= 0 && \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{E} &= 0, & \eta &= 0 && \text{on } \Gamma_D \end{aligned}$$

Primal Functional

$$\begin{aligned} G(\eta, \mathbf{E}, q; f, \mathbf{0}, 0) := & \left\| \frac{\mu}{\delta t} \eta + \nabla \times \mathbf{E} - f \right\|^2 \\ & + \left\| \nabla^\perp \eta - \sigma \mathbf{E} - \nabla q \right\|^2 + \left\| \nabla \cdot \sigma \mathbf{E} \right\|^2 \end{aligned}$$

Dual Problem

$$\mathcal{L}^* \mathcal{W} = \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} \eta_* \\ \mathbf{E}_* \\ q_* \end{pmatrix} = \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix} = \mathcal{U}$$

Dual Boundary Conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{E}_* &= 0, & q_* &= 0 && \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{E}_* &= 0, & \eta_* &= 0 && \text{on } \Gamma_D \end{aligned}$$

Dual Functional

$$\begin{aligned} G_*(\eta_*, \mathbf{E}_*, q_*; \eta, \mathbf{E}, q) := & \left\| \frac{\mu}{\delta t} \eta_* + \nabla \times \mathbf{E}_* - \eta \right\|^2 \\ & + \left\| \nabla^\perp \eta_* - \sigma \mathbf{E}_* - \sigma \nabla q_* - \mathbf{E} \right\|^2 + \left\| \nabla \cdot \mathbf{E}_* - q \right\|^2 \end{aligned}$$

Modified Boundary Conditions

Principle Idea

More boundary conditions on \mathcal{L} yield fewer boundary conditions on \mathcal{L}^*

Primal Boundary Conditions

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{E} = 0, \quad q = 0 & \quad \text{prescribed on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{E} = 0, \quad \eta = 0 & \quad \text{prescribed on } \Gamma_D\end{aligned}$$

Additional Boundary Conditions on \mathcal{L}

Boundary components: $\Gamma_{N_i}, \Gamma_{D_i}, i = 1, \dots, n$

Implied relationship: $\mathbf{n} \cdot \sigma \mathbf{E} = \mathbf{n} \cdot \nabla^\perp \eta = -\boldsymbol{\tau} \cdot \nabla \eta$

$$\int_{\Gamma_{N_i}} \mathbf{n} \cdot \sigma \mathbf{E} = - \int_{\Gamma_{N_i}} \boldsymbol{\tau} \cdot \nabla \eta = \eta(\Gamma_{D_i}) - \eta(\Gamma_{D_{i+1}}) = 0$$

Slack variable constraint: $q = 0$ everywhere

$$q = 0 \quad \text{on } \Gamma_D$$

Relaxed Boundary Conditions on \mathcal{L}^*

Dual Boundary Conditions

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{E}_* &= 0, & q_* &= C_i \quad \text{on } \Gamma_{N_i} \\ \eta_* &= 0 \quad \text{on } \Gamma_{D_i}\end{aligned}$$

\mathcal{L} no longer onto $\Leftrightarrow \mathcal{L}^*$ no longer 1-to-1

Existence of Solution in H^1 :

Define $W_* := \mathcal{D}^* \cap (H^1)^4$. If Γ_D not empty,

$$\mathcal{L}^* : W_* \rightarrow (L^2)^4 \quad \text{onto}$$

Solution not Unique:

For any solution $(\eta_*, \mathbf{E}_*, q_*) \in W_*$,

$$\mathcal{L} * \begin{pmatrix} \eta_* \\ \mathbf{E}_* \\ q_* \end{pmatrix} = \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix}$$

Numerical Solution Slows Down

Discrete Equations

Finite Element Spaces V_*^h , W_*^h and Z_*^h

Find $\eta_*^h \in V_*^h$, $\mathbf{E}_*^h \in W_*^h$ and $q_*^h \in Z_*^h$:

$$(\eta_*^h, \mathbf{E}_*^h, q_*^h) = \arg \min_{(V_*^h, W_*^h, Z_*^h)} G_*(\eta_*, \mathbf{E}_*, q_*; \eta, \mathbf{E}, q)$$

Dual Weak Problem

Find $\eta_*^h \in V_*^h$, $\mathbf{E}_*^h \in W_*^h$ and $q_*^h Z_*^h$:

$$\left\langle \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} \eta_*^h \\ \mathbf{E}_*^h \\ q_*^h \end{pmatrix}, \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} g_*^h \\ \mathbf{F}_*^h \\ p_*^h \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \frac{\mu}{\delta t} \eta_{old} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g_*^h \\ \mathbf{F}_*^h \\ p_*^h \end{pmatrix} \right\rangle$$

for every $g_*^h \in V_*^h$, $\mathbf{F}_*^h \in W_*^h$ and $p_*^h \in Z_*^h$.

Primal Variables

$$\begin{pmatrix} \eta^h \\ \mathbf{E}^h \\ q^h \end{pmatrix} = \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} \eta_*^h \\ \mathbf{E}_*^h \\ q_*^h \end{pmatrix}$$

Discrete Error Bounds

Dual Variables

$$\|\eta_* - \eta_*^h\|_1 + \|\mathbf{E}_* - \mathbf{E}_*^h\|_1 + \|q_* - q_*^h\|_1 \leq Ch^\beta (\|\eta_*\|_{1+\beta} = \|\mathbf{E}_*\|_{1+\beta} + \|q_*\|_{1+\beta})$$

Primal Variables

Symbolically

$$\|\mathcal{U} - \mathcal{U}^h\| = \|\mathcal{L}^* (\mathcal{W} - \mathcal{W}^h)\| \leq C \|\mathcal{W} - \mathcal{W}^h\|_1 \leq Ch^\beta \|\mathcal{W}\|_{1+\beta} \leq Ch^\beta \|\mathcal{U}\|_\beta$$

Detail

$$\begin{aligned} \left\| \begin{pmatrix} \eta - \eta^h \\ \mathbf{E} - \mathbf{E}^h \\ q - q^h \end{pmatrix} \right\| &= \left\| \begin{bmatrix} \frac{\mu}{\delta t} & \nabla \times & 0 \\ \nabla^\perp & -\sigma & -\sigma \nabla \\ 0 & \nabla \cdot & 0 \end{bmatrix} \begin{pmatrix} \eta_* - \eta_*^h \\ \mathbf{E}_* - \mathbf{E}_*^h \\ q_* - q_*^h \end{pmatrix} \right\| \leq C \left\| \begin{pmatrix} \eta_* - \eta_*^h \\ \mathbf{E}_* - \mathbf{E}_*^h \\ q_* - q_*^h \end{pmatrix} \right\|_1 \\ &\leq Ch^\beta \left\| \begin{pmatrix} \eta_* \\ \mathbf{E}_* \\ q_* \end{pmatrix} \right\|_{1+\beta} \leq Ch^\beta \left\| \begin{pmatrix} \eta \\ \mathbf{E} \\ q \end{pmatrix} \right\|_\beta \end{aligned}$$

Numerical Results

- Steady State
- Discontinuous Coefficients ($\sigma_0 = 1, \sigma_1 = 6.33 \cdot 10^7$)
- Irregular Boundary Conditions
- $H_{DC_\sigma} = H(\operatorname{div}, \sigma) \cap H(\operatorname{curl}) \neq (H_1)^2$

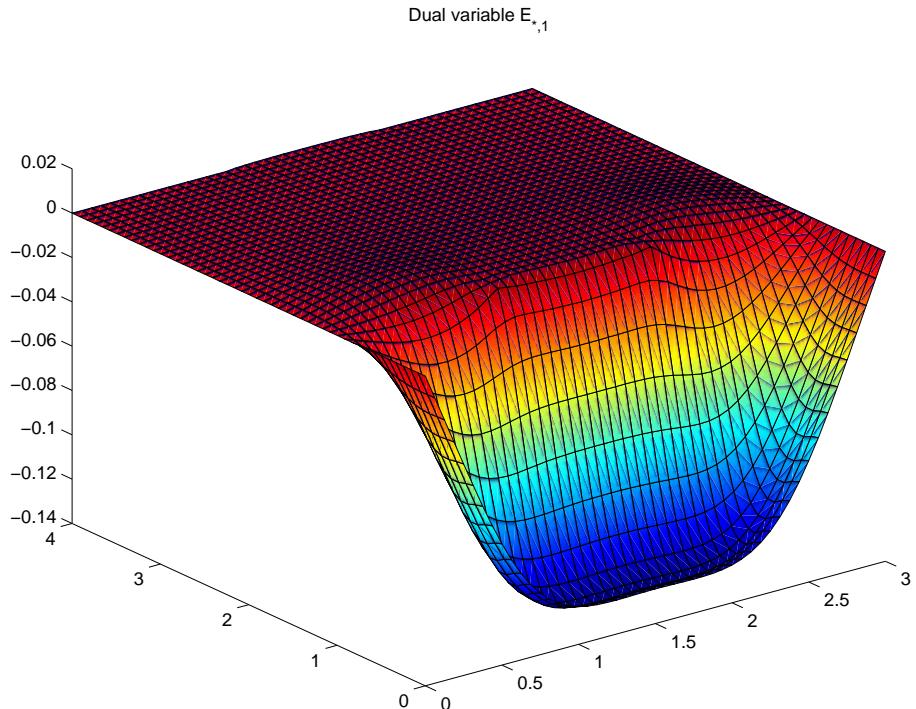
Convergence results :

L/h	MG(V)	MG(W)	MG(CG)	$G_*(\eta_*^h, \mathbf{E}_*^h)$	Reduction
12	0.99	0.99	0.65	0.15449	$2^{-2} = 0.25$
24	0.97	0.98	0.62	0.05699	0.3689
48	0.96	0.96	0.44	0.01680	0.2948
96	0.95	0.90	0.24	0.00389	0.2315
192	0.95	0.81	0.14	0.00000	

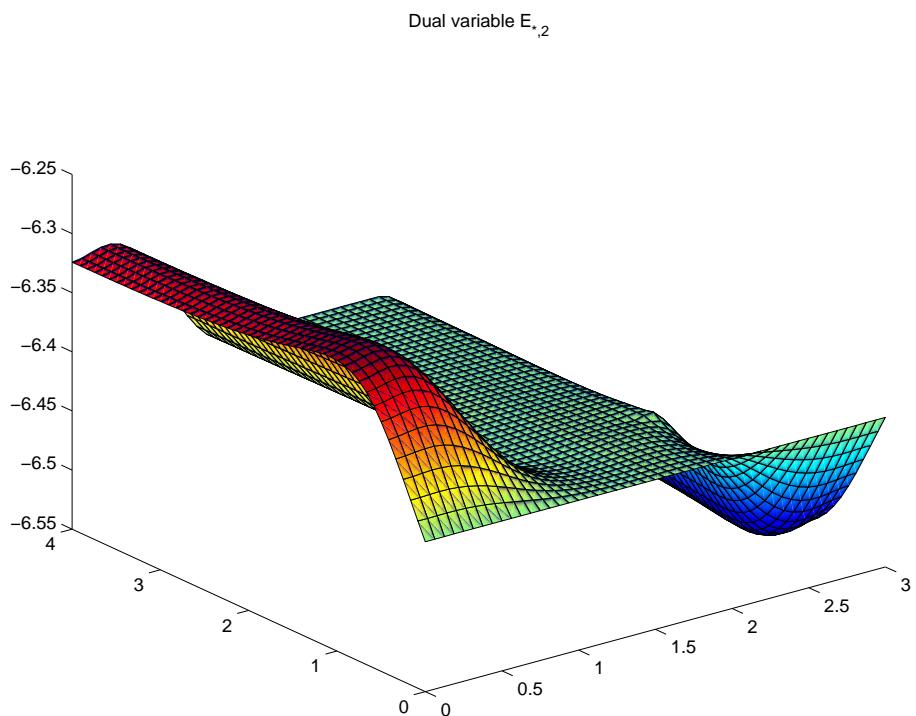
- Rectangular Mesh
- Bilinear Finite-elements
- Multigrid V(1,1) and W(1,1)-Cycle
- Conjugate Gradient Acceleration

Numerical Results

Dual variable e_{1*}^h :

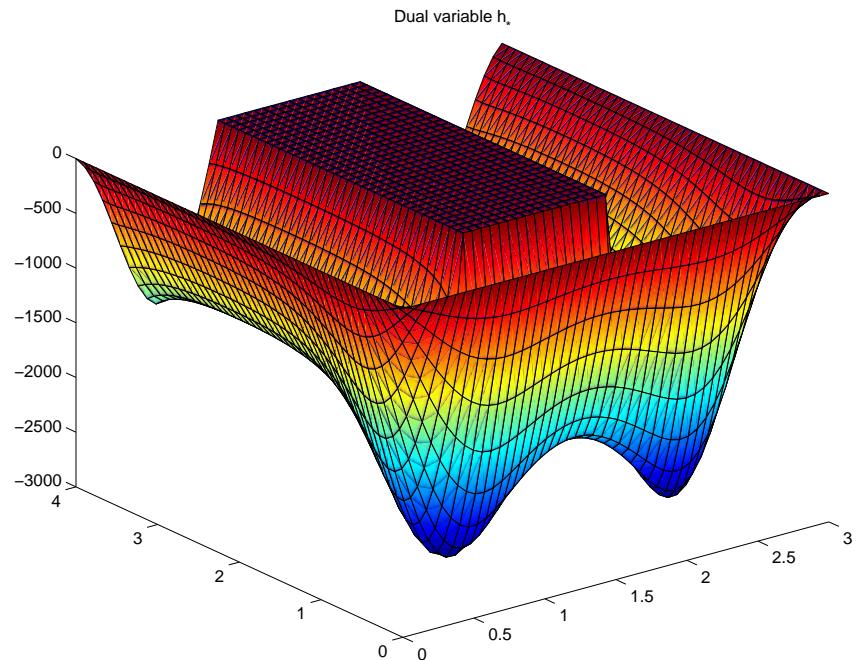


Dual variable e_{2*}^h :

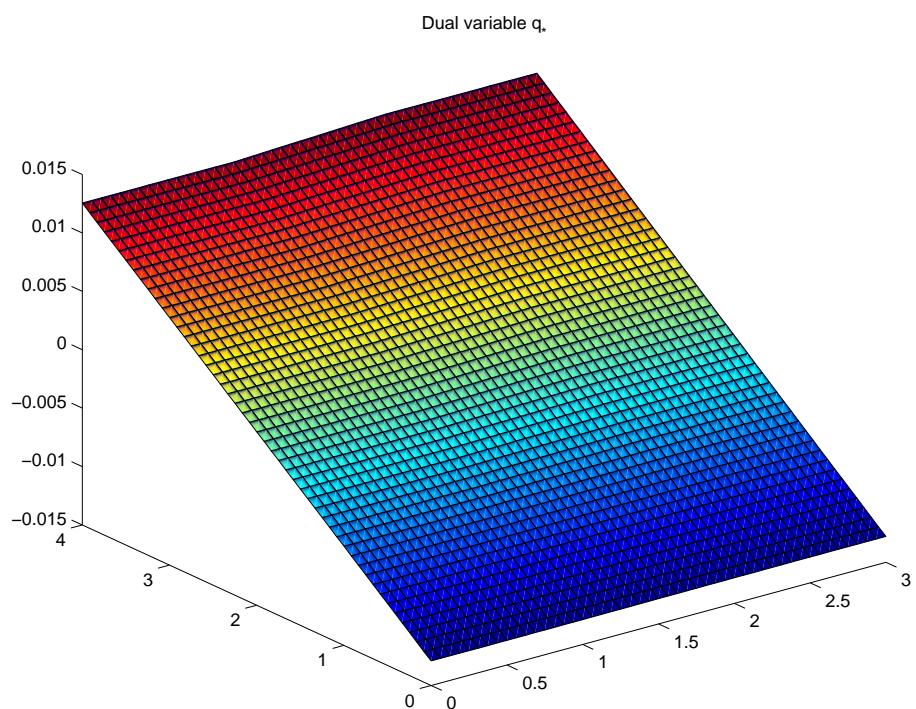


Numerical Results

Dual variable η_*^h :

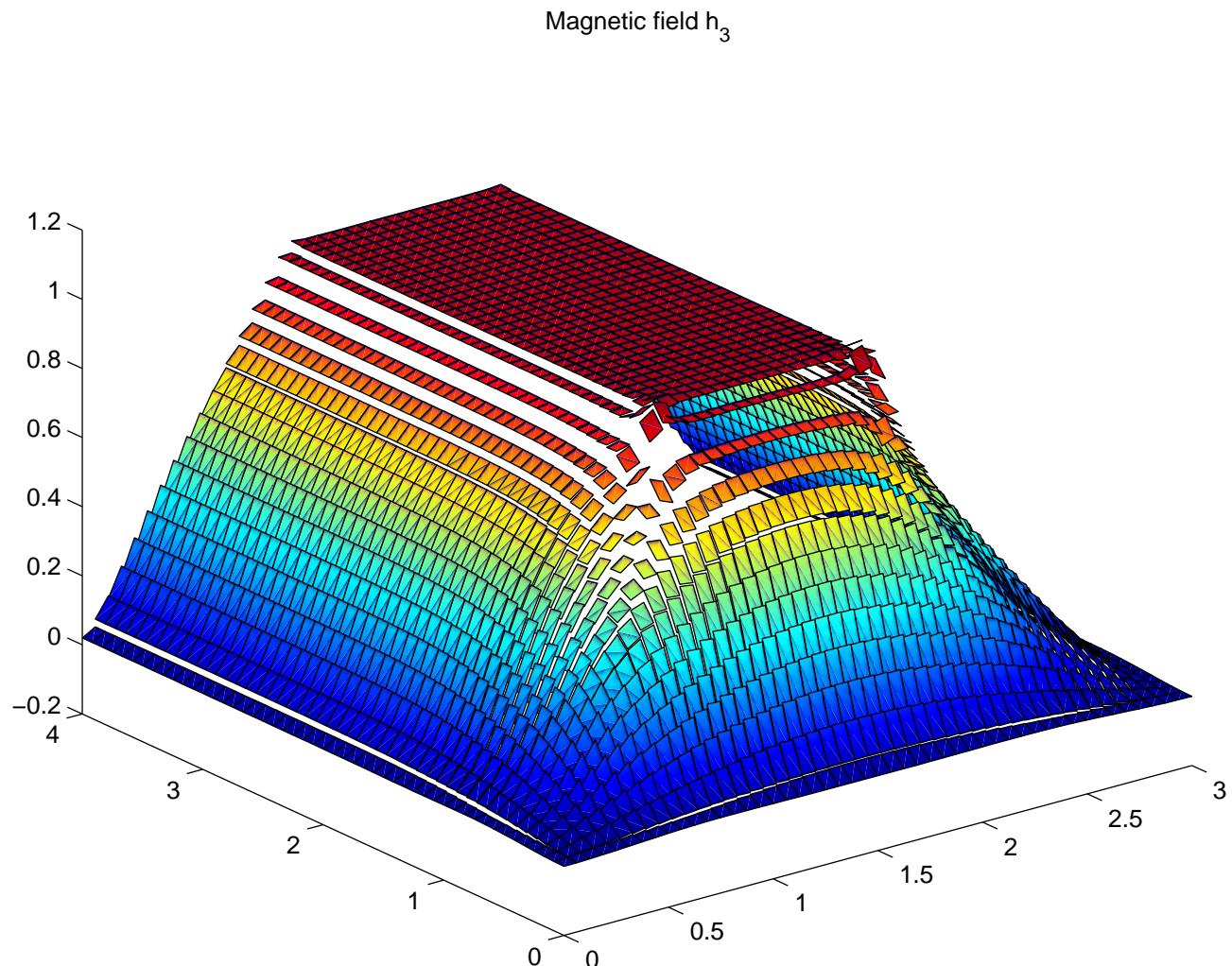


Dual variable q_*^h :



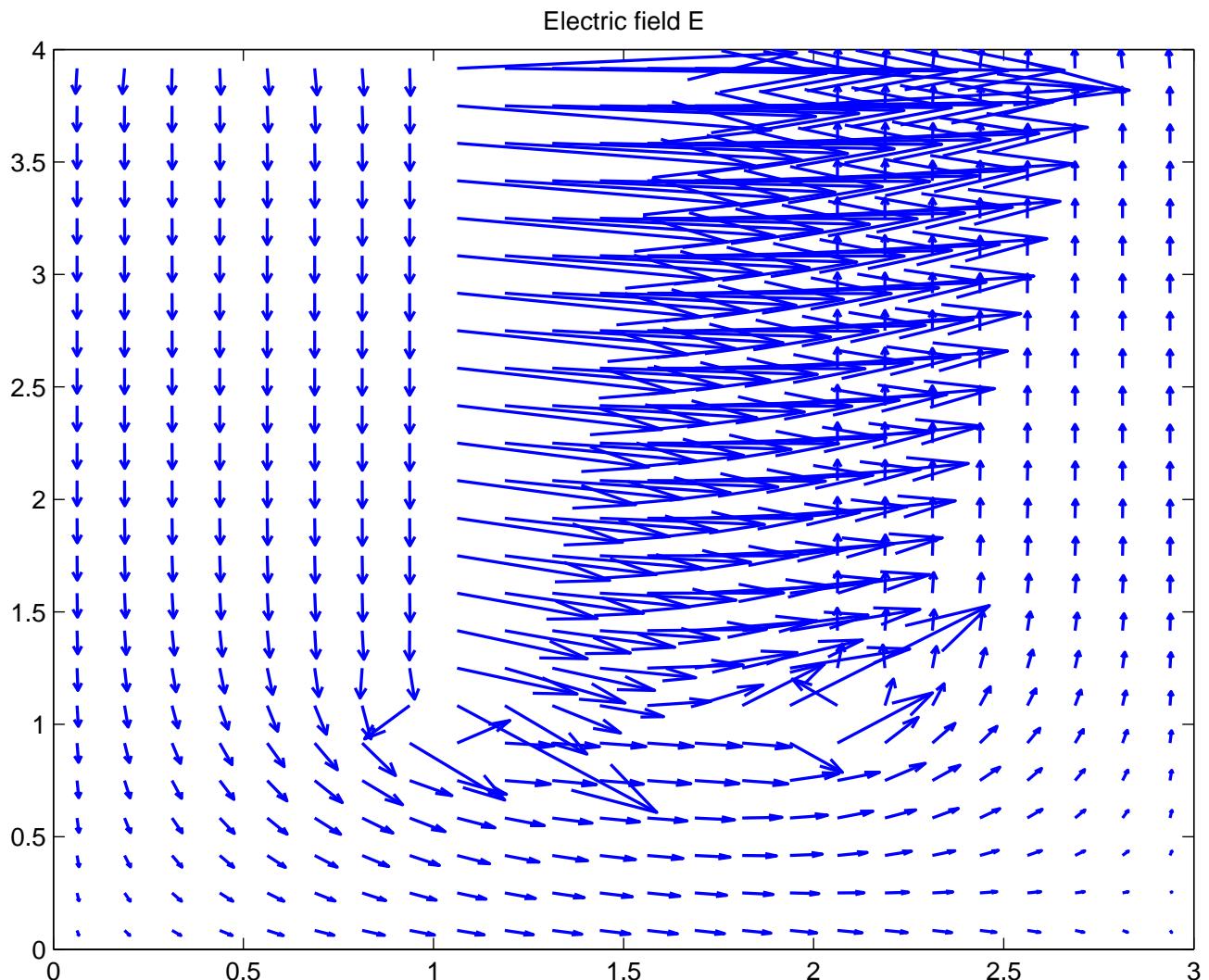
Numerical Results

Magnetic field η^h :



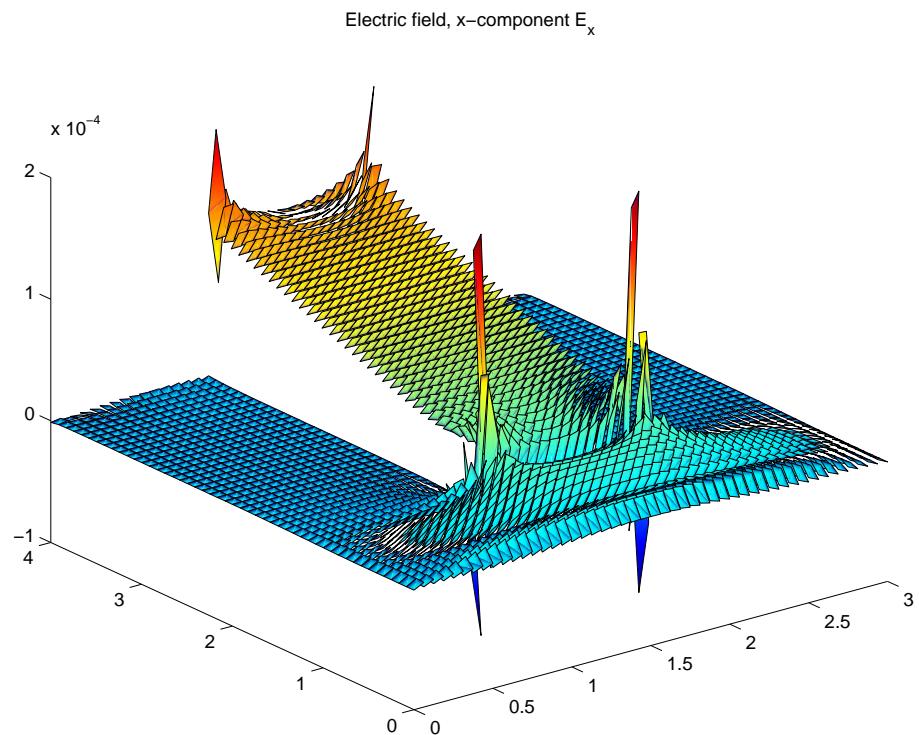
Numerical Results

Electric field \mathbf{E}^h :

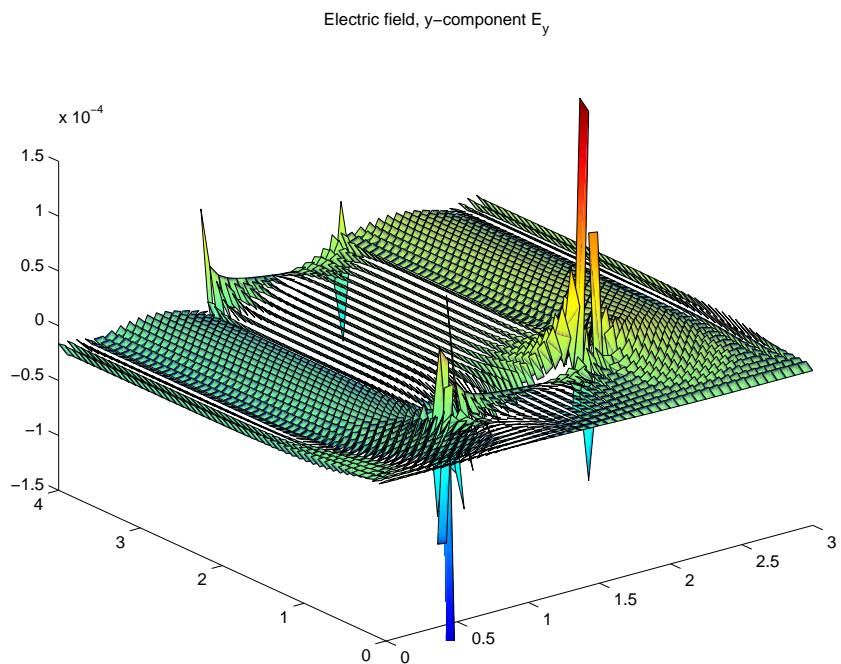


Numerical Results

Electric Field x-component e_1^h :

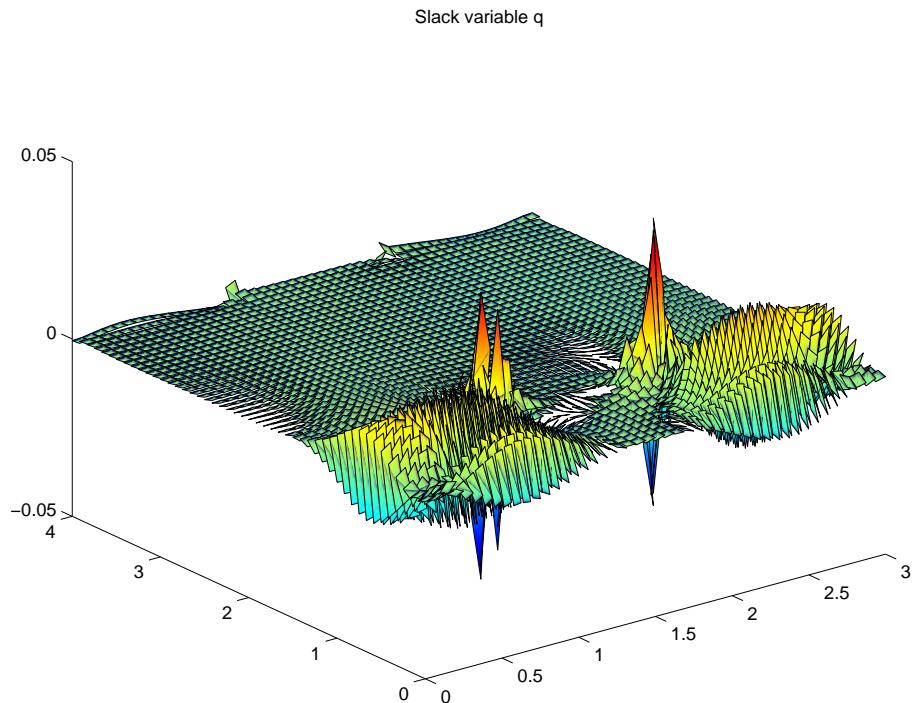


Electric Field y-component e_2^h :



Numerical Results

Slack variable q^h :



Summary

Maxwell's Equations in 2-D

- FOSLL Formulation

- Smooth Coefficients and Regular Boundary Conditions
 - * Optimal Finite-Element Convergence in H^1
 - * Enhanced Finite-Element Convergence in L^2
 - * Optimal Multigrid Performance
- Discontinuous Coefficients or Irregular Boundary Conditions
 - * Solutions in $H_{DC} = H(\text{div}, \sigma) \cap H(\text{curl}) \neq (H^1)^2$
 - * Several Alternatives Include FOSLL*

- FOSLL* Formulation

- Discontinuous Coefficients Straightforward
- Irregular Boundary Conditions More Difficult
- Finite-Element Convergence
 - * Dual Variables Optimal in H^1
 - * Primal Variables Optimal in L^2
- Optimal Multigrid Performance